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A FRAMEWORK FOR UNDERGRADUATE STUDENTS' MATHEMATICAL FORESIGHT

Wes Maciejewski and Bill Barton

The University of Auckland

Problem solving has become a central topic in mathematics education at all levels of schooling. Despite this, there is much left to be understood about student problem solving activity. In a previous paper (Maciejewski & Barton, under review) we introduced the notion of mathematical foresight to characterize mathematicians' initial approaches to research-level mathematical situations. This paper extends this previous theoretical work through a qualitative, empirical analysis of student problem solving through the lens of mathematical foresight. In so doing, we generate a framework of student pre-planning activity during problem solving. It is hoped that our approach of analyzing one aspect of problem solving activity in detail will enrich our understanding of the problem solving process as a whole.

INTRODUCTION

When encountering a new mathematical situation, students often struggle to make initial progress. This observation has been the impetus for a significant amount of research in mathematics education. Much of this research has appeared in the context of *problem solving*. A mathematical problem is, following Schoenfeld's (1983) description,

> ...only a problem (as mathematicians use the word) if you don't know how to go about solving it. A problem that has no 'surprises' in store, and can be solved comfortably by routine or familiar procedures (no matter how difficult!) is an exercise. (Schoenfeld, 1983; p. 41).

This suggests that the defining feature of a mathematical problem is not intrinsic to the problem; rather, it is how the problem relates to the solver. A too-familiar mathematical situation may cause its solver to act with little thought, as may be the case with elementary addition for an undergraduate student, whereas a completely unfamiliar situation may cause its solver to not act at all, as in the case of a question in modular forms posed to a high school student. Problems are those mathematical situations not too familiar, but still understandable to the solver. This current work is motivated by a desire to better understand how students might act in such situations.

A number of frameworks for effective problem solving have been identified: Schoenfeld's (1985) notions of *resources*, *control*, *heuristics*, and *beliefs*, and Pólya's *understanding the problem*, *devising a plan*, *carrying out the plan*, and *looking back*, for examples, among others (English & Sriraman, 2010). An important activity common to all of these problem-solving frameworks is *planning*: the identification of the steps to be taken by the student when constructing their solution. We propose that students engage in activity prior to planning a solution, one that is analogous to what we call *mathematical foresight*.

As formulated in (Maciejewski & Barton, *under review*), mathematical foresight is the ability to see a likely shape of a resolution to a mathematical situation—a term we use to encompass problem situations, but also more general situations, such as *modelling situations* (Lesh & Zawojewski, 2010)—and a, possibly wide, trajectory that will lead to that resolution. This act of foresight is performed upon initial contact with the new mathematical situation. It is an active and on-going process wherein the solver is able to provide some kind of justification for their imagined resolution and path, although it may not be mathematically rigorous. Foresight is more than intuiting an answer—we take intuition to be an unconscious, immediate feeling of conviction—and foresight is problem-specific, which distinguishes it from general, heuristic problem solving strategies. A cartoon of our model of mathematical foresight is presented in Figure 1.

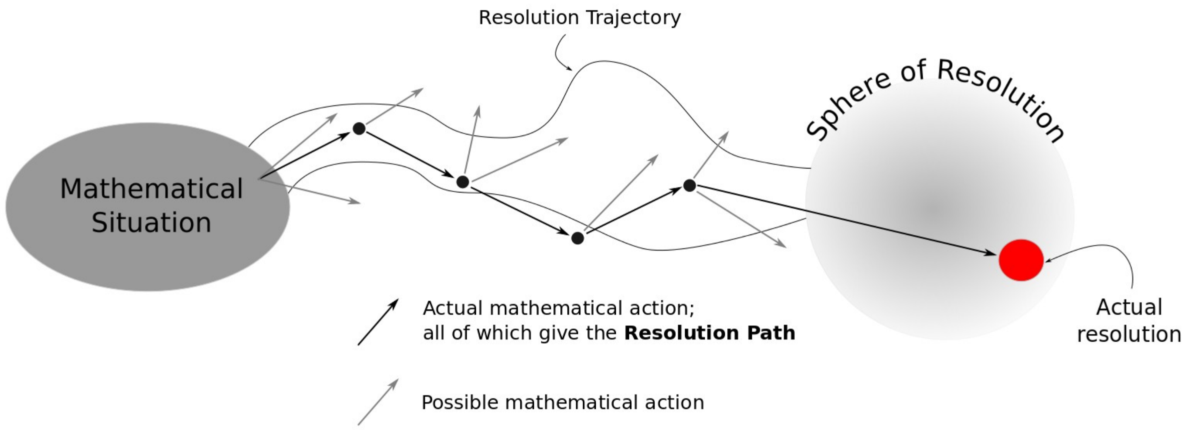


Figure 1: *Mathematical foresight is the process of imagining a resolution to a mathematical situation (the sphere of resolution) and a path likely to lead to that resolution (the resolution trajectory). The figure is reproduced with permission from (Maciejewski & Barton, under review).*

Results from a series of interviews, reported in (Maciejewski & Barton, *under review*), indicate that mathematicians identify mathematical foresight as an activity central to their research activity, and one in which they engage regularly. For example, they use foresight to help them choose which research problems to tackle and which to leave alone. We propose that the activity of a student working a problem is analogous to that of the mathematician working at the research level: both are working in novel-to-them situations and have their knowledge and experience to bring to bear. Mathematical foresight, we hypothesize, therefore ought to be observed in student problem-solving activity.

The current article presents a descriptive framework of student thinking emerging from initial contact with mathematical situations through the lens of mathematical foresight. The framework was constructed from data gathered in two rounds of student responses to five mathematical tasks. The results reported here aid in refining the model of mathematical foresight and highlight its potential importance in mathematics education.

Task 1: A Modelling Task

We are becoming familiar with the sight of algal blooms on our beaches in the Summer. We are told that these are more frequent as a result of warmer sea temperatures and nutrient concentrations. We observe that the bloom congregates in certain places on our beaches, and comes and goes over a period of days and weeks.

How might we construct a mathematical model of the algal bloom? What possible forms might the model take? Given a certain type of model, what might the variables be and how would they be represented? What mathematical characteristics is the model likely to have?

Task 2: A Graphing Task

Graph a function $h(x)$ that satisfies the following conditions.

- $h(0) = 2$
- $h'(x) > 0$ when $x < -1$, $h'(x) < 0$ when $x > -1$
- $h''(x) > 0$ when $x < -2$ and when $x > 0$, $h''(x) < 0$ when $-2 < x < 0$
- $\lim_{x \to 0} h'(x) = \infty$

Draw a graph without performing too much written work. Justify your choice of this graph by writing your thoughts in as much detail as possible.

Task 3: A Mathematical Game

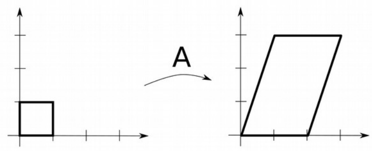
Suppose you are playing a mathematical game with a group of people. The game is as follows. Everyone writes down a number between 0 and 100, inclusive. The average is calculated. Each person's score is calculated to be the distance between their score and two-thirds the average. Mathematically,

$$Score = |Number - 2/3(Average)|$$

Your goal is to get as low a score as possible. What number do you write down?
Justify your choice of this number by writing your thoughts in as much detail as possible.

Task 4: An Eigenvalue Task

Suppose the matrix A maps the unit square in $\mathbb{R}^2$ to the parallelogram in the right half of the figure below.



What are the eigenvalues of A^{-1}?

Task 5: A Volume Task

Suppose you work on a road construction site where fuel for all the machines is stored in a large cylindrical container. This cylinder is lying on its side. There is a small opening halfway along the tank facing upwards in which a measuring stick can be inserted to find the distance between the surface of fuel and the opening. What is an expression for the amount of fuel in the cylinder?

Figure 2: *The five mathematical situations used in this study.*

METHODS

The data used in this study was gathered in two independent sessions. In the first, students in a third-year undergraduate mathematics course intended for prospective teachers were presented with tasks one to three in Figure 2 and asked to write their approaches to solving each. The second round consisted of a set of interviews with 11 student volunteers, each enrolled in a first-year mathematics course covering calculus and linear algebra and who did not participate in the first round. These students were given two of Task 1, 2, 4, or 5—task three was excluded from this round of data gathering, based on the poor responses to the task given by the students in round one—asked to think about how they would solve the task, and interviewed about their imagined approaches to a solution. Tasks four and five were created for use in the second round based on the researchers' perceived need for tasks that appeared familiar to the students. The interviews were recorded and transcribed.

The tasks in Figure 2 were created with the intention of encompassing a variety of mathematical situations, from (1) an applied modelling task where the student is asked to describe an authentic situation with mathematics, (2) one in which the student is asked to generate a graph of a function given only properties of the function, (3) a task from game theory where the student must choose and justify a strategy, (4) one on linear transformations, and (5) another asking for the volume of an uncommon shape. Initial student participants were given a few minutes to describe what they would do with each problem, or how they saw it, without actually attempting to solve the problem. From this first round of participant responses we generated a draft of the framework below. This was achieved by each author individually identifying features of each solution that were indicative of foresighting behaviour, and generating a classification for these features. We then met and created the framework from our separate observations. Next, we returned to the data and separately re-interpreted it in terms of categories of the framework. We then met to compare the framework categories we individually assigned to each student response to check for agreement and to make any necessary revisions. The focus of our attention in the analysis of the first round of data was the anticipated solution trajectory rather than the shape of the final solution, although some students did comment on the solution space. The second round of data aided in supporting our initial framework and generated a more detailed classification of students' imagining of the sphere of resolution.

Below is the framework describing students' initial problem-solving thoughts that emerged from an analysis of the students' utterances and inscriptions through the lens of mathematical foresight. Recall that there are two components of mathematical foresight: forming an image of (i) the resolution to the mathematical situation (the *sphere of resolution*), and (ii) a likely path to that resolution (the *resolution trajectory*). Both of these can be imagined to varying degrees of clarity. In addition, clarity with one of these does not necessitate clarity in the other: a student may see the form of the resolution but be no closer to reaching the resolution. Both of these components were considered during the analysis of the student inscriptions and utterances.

RESULTS

We consider the sphere of resolution and resolution trajectory separately, in turn. Though we recognize that these two may not exist as separate entities in the students' minds, we find we are in need of a way of analyzing these two simultaneously. Until our methodology has been developed to allow us to do so, we keep the analyses separated. We then consider possible interactions between students' images of the sphere of resolution and resolution trajectory.

Sphere of resolution

Responses, both written and spoken, indicated that students were often able to imagine a resolution to the mathematical task. We characterize these responses as four qualitatively-distinct categories, arranged according to increasing clarity of the imagined resolution.

1. *No image of the resolution.* Student responses in this category had no indication of a resolution to the mathematical task.
2. *A generic image.* Responses of this type were general statements concerning the nature of the resolution to the given task. For example, some students indicated the resolution to Task 1 is a system of differential equations but were unable to elaborate.
3. *An incomplete image.* Some students indicated particular features of their imagined resolution, which was not completely well-formed. For example, in addition to imagining a resolution to Task 1 as being a system of differential equations, some students indicated the system must be periodic.
4. *A particular image.* Responses in this category are explicit forms for a resolution. For example, one student wrote "|Sin(ax+b)|" as a particular resolution to Task 1.

Resolution Trajectory

Students' imagined resolution trajectories were more nuanced than their images of the spheres of resolution. This is due to resolution trajectories having more *degrees of freedom*; trajectories can be imagined with varying degrees of clarity, as was the case above, but there is more freedom in how the trajectories are imagined. We were nevertheless able to construct a descriptive framework of the students' images of the resolution trajectories, presented below. The levels of the framework each have varying degrees of clarity and we take higher levels as subsuming lower levels.

1. *No indication of foresight*. Responses of this type had little relevance to the problem with no indication of anticipated progress.
2. *Identifying factors relevant to the resolution*. Students at this level are able to identify information that is given either explicitly or implicitly in the problem statement that will aid in its solution. In the Graphing task, many students identified what properties of the function affect the shape of the graph; e.g. the first derivative conditions lead to where the graph is increasing/decreasing. This level is distinguished from level 0 in at least one important regard: the control exhibited by the student, as in the sense of (Schoenfeld, 1985). Students identify relevant factors but also identify factors *not* relevant either explicitly or by refraining from writing them.
3. *Creating/identifying (mathematical) relationships, between the relevant factors*. At this level, students are able to recognize how relevant factors (ought to) interact to contribute to a resolution. These interactions may or may not be explicitly mathematical. In the Mathematical Game task, many students identified that a strategy must consider possible actions of the other players.
4. *Recognizing consequences of the relationships*. Having established how the factors relate, a student at this level identifies the mathematical consequences of these relationships. For example, one student identified algal growth as a relevant factor in the modelling task and chose to represent the relationship between algal concentration and time as exponential.

They then write, “would likely see a curve as conditions approach the ideal.” The connection between this statement and the exponential relationship is not entirely clear, but we suspect the student is anticipating a sigmoidal, logistic relationship, which can involve an exponential function, between algal concentration and time, with concentration levelling out as saturation is approached.

5. *Identifying limitations/strengths/generalizations of the chosen approach*. Responses at this level were exhibited by only a couple of participants. One participant was to write out a complete expression for the volume requested in Task 5, which involved the radius r of the cylinder as a parameter and the length l of the measuring stick as a variable. He verbalized how there ought to be two formulas, one for $l < r$ and the other for $l > r$. He then realised that this was irrelevant if the equation was set up in a certain way: “And here it doesn't matter, 'cause it's all squared...so it doesn't make a difference.”

Interactions between resolution and trajectory

There is some indication from the data that a student's ability to imagine the resolution or trajectory is related to their ability to imagine the other. Such a relationship is expected, given what is known of the mathematical foresight of working mathematicians (Maciejewski & Barton, *under review*). Only one participant demonstrated strength in imagining both the sphere of resolution and trajectory.

The bi-directional relationship mentioned above did not consistently exist in the student responses. Some could imagine a particular form of a resolution but had no clear indication of a trajectory. Others could imagine a trajectory without a clear sphere of resolution. For example, one student solving Task 5 could not see a possible form for the volume expression, but suspected it could be arrived at by using the formula for the volume of a cylinder: “The general form … is an equation … Yep, it's blank. How I would go about finding the solution, I'd chuck in the cylinder equation.” They continue by identifying the length and radius of the tank as being important but are unable to incorporate the height from the top of the fuel to the top of the cylinder.

Moreover, the influence of one dimension on the other could work either way: a student may imagine a resolution and this may inform their image of the trajectory, or a focus on particular features of the task and ways to set out on the trajectory may inform an image of the resolution.

DISCUSSION

At the outset, we suspected students might engage in mathematical foresight when encountering a novel mathematical situation. This was supported by data from participants in our two-round study reported here. The data gathered has informed the creation of a framework that describes students' initial thinking about a mathematical situation through the lens of mathematical foresight.

The framework presented here has elaborated our initial model for mathematical foresight as presented in (Maciejewski & Barton, *under review*). The mathematical foresight exhibited by mathematicians sees the two components, the sphere of resolution and the resolution trajectory, as coupled: one does not exist without the other. This was not true for the students who participated in this present study. Some students were able to see a likely form for a resolution to a mathematical task but were unable to see a trajectory leading to that resolution. Others could see how to “set out” on a resolution trajectory without seeing a resolution. This has led to a refinement of our model for mathematical foresight by viewing the sphere of resolution and resolution trajectory as two *dimensions* of mathematical foresight. Rather than co-existing, each can exist to varying degrees of clarity, including not existing at all. We see a great need for a better understanding of how these two dimensions interact. Our initial thoughts were that with greater clarity in one dimension comes greater clarity in the other. Considering this was not necessarily true for our participants, we are left wondering why.

Mathematical foresight is one activity in which a student might engage when encountering a mathematical problem situation. We view this activity as preceding solution planning and informing what plan is ultimately made. We also expect beneficial interactions between students' mathematical foresight and other aspects of successful problem-solving behaviour. For example, being able to foresight in a given mathematical situation may lead to improved persistence, since a hypothetical path is laid before embarking, and greater control, in the sense of Schoenfeld (1985).

In the interviews we conducted for this study, students often verbalized personal experiences of learning and working with mathematics. We expect that these personal experiences both enable and limit the ability of the students to engage with novel mathematical situations. This observation has led us to wonder about the similarities between mathematical foresight and foresight in common, non-mathematical, planning situations. A recent body of work in cognitive psychology indicates that *episodic memories*—memories of personally-experienced events—aid in planning future, to-be-experienced events (Atance & O'Neill, 2001; Schacter, Addis, & Buckner, 2007). This is surprising for situations that appear to best be planned for *semantically*; that is, with factual memory. We are left wondering about the influence of episodic memories of working with mathematics on problem solving behaviour. A companion manuscript (Maciejewski, Roberts, & Addis, *under review*) presents a categorization of students' episodic future thinking but does not examine how such thinking relates to the students' ability to engage with the mathematical situation. A closer examination of how these relate would deepen our understanding of students' authentic planning thoughts and behaviours in mathematical situations.

The next step to the development of the framework presented here is to collect further student written and verbal responses to revised versions of the tasks. We expect such data to aid in refining the categories, but also to further elucidate the role of foresight in student problem solving. We imagine mathematical foresight as a dynamic process, evolving as the user of mathematics engages more fully with the situation. We expect, then, that a student's mathematical foresight is more than just a point in the two-dimensional space formed by resolution trajectory and sphere of resolution axes. Rather, the student forges a path through this space over time as they work in the mathematical situation. With an increase in clarity for the imagined sphere of resolution may come increased clarity in the path that leads to that resolution and, possibly, vice versa.

We view the strength of this work as being a re-casting of problem-solving as active and evolving, comprising action and anticipation. Rather than view effective problem-solving as the ability to select and bring to bear relevant knowledge and heuristics, we propose that problem solvers have degrees of knowledge of how a resolution to a mathematical situation will evolve during the solution process and this *foresight* guides their enacted actions. Considering this is a novel perspective, as far as our knowledge of the literature indicates, there are a number of directions for future research. The most important for us is addressing the question, are those more able to foresight *better* (faster/accurate/creative) problem solvers? Throughout this work, we purposely did not evaluate the correctness, or even the appropriateness, of the students' proposed resolutions. Indeed, a student could possibly form a completely lucid image of an incorrect resolution. But is this possible if their image of the resolution trajectory is equally as lucid?

Another direction for research is to investigate the role of mathematical foresight in managing self-regulation while working with mathematics. Previous work has shown that not all forms of mental simulation are beneficial or productive. This is clearly true for ruminations on traumatic events, but surprisingly also true for simulations with an exclusive focus on the outcome of a situation. Students who imagine a process required to reach a desired goal—imagining studying to achieve a high grade, for example—are better able to manage affective psychological factors and exhibit greater self-regulation than those who strictly focus on attaining the goal (Taylor, Pham, Rivkin, & Armor, 1998). Therefore, we suspect students who strengthen their abilities in both dimensions of mathematical foresight to be less hindered by negative emotions and better able to control their actions in mathematical situations.

Finally, we return to how we began—a problem is only a problem if it relates to the solver in that way. But how might a problem relate to a solver? We know that this relationship is more than binary—either the student knows how to solve the problem or they do not yet—and a holistic description is not yet found in the literature. A mathematical situation can evoke a range of reactions, including forming images of possible resolutions, and we suggest these need to be considered when documenting students' authentic problem-solving behaviour. Furthermore, an educator ought to assist their students in making these reactions mathematically productive. Fostering mathematical foresight may be a means to this.

REFERENCES

Atance, C. & O'Neill, D. (2001). Episodic future thinking. *Trends in Cognitive Neuroscience, 5*, 533-539.

English, L. & Sriraman, B. (2010). Problem solving for the 21st Century. In Sriraman, B. & English, L. (Eds.), *Theories of mathematics education: Seeking new frontiers, Vol. 1* (pp. 263-290). Springer: Berlin.

Lesh, R. & Zawojewski, J. S. (2007). Problem solving and modelling. In F. Lester (Ed.), *The Second Handbook of Research on Mathematics Teaching and Learning* (763–804). Charlotte, NC: Information Age Publishing.

Maciejewski, W. & Barton, B. (*under review*) Mathematical foresight: towards a description.

Maciejewski, W., Roberts, R., & Addis, D.R. (*under review*). Episodic future thinking in mathematical situations.

Pólya, G. (1945). *How to Solve It*. Garden City, NY: Doubleday.

Schacter, D.L., Addis, D.R., & Buckner, R.L. (2007). Remembering the past to imagine the future: the prospective brain. *Nature Reviews Neuroscience, 8*, 657-661

Schoenfeld, A. (1983). The wild, wild, wild, wild, wild world of problem solving: A review of sorts. *For the Learning of Mathematics, 3*, 40-47.

Schoenfeld, A. (1985). *Mathematical Problem Solving*. London, UK: Academic Press.

Taylor, S.E., Pham, L.B., Rivkin, I.D., & Armor, D.A. (1998). Harnessing the imagination. Mental simulation, self-regulation, and coping. *American Psychologist, 53, 4*, 429-39.